Now that we know how to calculate force, let's compute the Coulomb force:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{4} F_{\mu\nu} F_{\delta\lambda} g^{\delta\mu} g^{\lambda\nu}$$

$(\partial_0 A_i)^2$ term has to have positive sign!

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $g_{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$

→ add mass for photon (send to zero later):

$$\mathcal{L}' = \mathcal{L} + \frac{1}{2} m^2 A_\mu A^\mu + A_\mu J^\mu$$

current

assume $\partial_\mu J^\mu = 0$
"current is conserved"

→ $Z[J] = \int \mathcal{D}A \, e^{iS(A)} = e^{iW[J]}$

where

$$S(A) = \int d^4x \, \mathcal{L}'$$
$$= \int d^4x \left(\frac{1}{2} A_\mu \left[(\partial^2 + m^2) g^{\mu\nu} - \partial^\mu \partial^\nu \right] A_\nu + A_\mu J^\mu \right)$$

have integrated by parts

→ have to solve

(*) $\left[(\partial^2 + m^2) g^{\mu\nu} - \partial^\mu \partial^\nu \right] D_{\nu\lambda}(x) = \delta_\lambda^\mu \delta^{(4)}(x)$

Define momentum space propagator:

$$D_{\nu\lambda}(x) = \int \frac{d^4k}{(2\pi)^4} D_{\nu\lambda}(k) e^{ik\cdot x}$$

Plugging into (*), we find

$$\left[-(k^2 - m^2) g^{\mu\nu} + k^\mu k^\nu\right] D_{\nu\lambda}(k) = \delta_\lambda^\mu$$

$$\longrightarrow D_{\nu\lambda}(k) = \frac{-g_{\nu\lambda} + k_\nu k_\lambda / m^2}{k^2 - m^2}$$ "massive meson propagator"

Alternatively, use

$$D_{\nu\lambda}(x) = \langle 0 | T\{A_\nu(0) A_\lambda(x)\} | 0 \rangle$$

Lecture 2

polarization

$$A_\mu(x) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}} \sum_{\sigma=1}^{3} \left[e^{ik\cdot x} \varepsilon_\mu^{(\sigma)}(k)\, a(k) + e^{-ik\cdot x} \varepsilon_\mu^{(\sigma)*} a^\dagger(k) \right]$$

with "Lorentz gauge": $\partial_\mu A^\mu = 0 \rightarrow k^\mu \varepsilon_\mu = 0$

$\longrightarrow$ 3 linearly independent polarizations

Then $D_{\nu\lambda}(k) = \dfrac{\sum_\sigma \varepsilon_\nu^{(\sigma)}(k)\, \varepsilon_\lambda^{(\sigma)*}(k)}{k^2 - m^2 + i\varepsilon}$

$k^\mu \varepsilon_\mu = 0 \longrightarrow k^\nu D_{\nu\lambda} \overset{!}{=} 0$

$\longrightarrow \sum_\sigma \varepsilon_\nu^{(\sigma)} \varepsilon_\lambda^{(\sigma)*} \sim -g_{\nu\lambda} + \dfrac{k_\nu k_\lambda}{m^2}$ unique solution up to positive proportionality const.!

Now $W[J] = -\frac{1}{2}\int \frac{d^4k}{(2\pi)^4} J^\mu(k)^* \frac{-g_{\mu\nu} + k_\mu k_\nu/m^2}{k^2 - m^2 + i\varepsilon} J^\nu(k)$

$0 = \partial_\mu J^\mu \longleftrightarrow k_\mu J^\mu = 0$

$\rightarrow W[J] = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4} J^\mu(k)^* \frac{1}{k^2 - m^2 + i\varepsilon} J_\mu(k)$

We see that the sign is opposite to the φ- theory from last lecture!

$$E = \frac{1}{4\pi r} e^{-mr}$$

now send $m \rightarrow 0 \Rightarrow E \rightarrow \frac{1}{4\pi r}$

$$\rightarrow \vec{F} = \vec{\nabla} E = -\frac{\vec{e}_r}{4\pi r^2}$$

"repulsive force, decreasing quadratically with distance"

To accommodate positive/negative charges, we write $J^\mu = J_p^\mu - J_n^\mu$

$\rightarrow$ lump with charge density J_p^0 is attracted to lump with charge density J_n^0

Let us now look into gravity!

→ described by spin 2 particle

$h_{\mu\nu}(x) \longrightarrow D_{\mu\nu,\rho\lambda}(k) = \dfrac{\sum_\sigma \varepsilon^{(\sigma)}_{\mu\nu}(k)\, \varepsilon^{(\sigma)}_{\rho\lambda}(k)^*}{k^2 - m^2}$

characterized by 5 polarization tensors

$\varepsilon^{(a)}_{\mu\nu}\ (a = 1, 2, \ldots, 5)$

satisfying $\quad \underset{(a)}{k^\mu \varepsilon^{(a)}_{\mu\nu} = 0}\ , \quad \underset{(b)}{g^{\mu\nu} \varepsilon^{(a)}_{\mu\nu} = 0}$

(gauge conditions)

graviton $\varepsilon_{\mu\nu}$ arises from perturbations of metric of spacetime (general relativity)

→ symmetric Lorentz tensor,

has $4 \cdot 5/2 = 10$ components

(a) → 4 conditions

(b) → 1 condition

→ $10 - 4 - 1 = 5$ independent comp.

→ solution:

$$\sum_a \varepsilon^{(a)}_{\mu\nu}(K)\, \varepsilon^{(a)}_{\lambda\sigma}(K) = (G_{\mu\nu} G_{\nu\sigma} + G_{\mu\sigma} G_{\nu\lambda}) - \frac{2}{3} G_{\mu\nu} G_{\lambda\sigma}$$

(exercise)

Thus we compute

$W(T)$

$$= -\frac{1}{2}\int \frac{d^4k}{(2\pi)^4} T^{\mu\nu}(k)^* \frac{(G_{\mu\lambda}G_{\nu\sigma} + G_{\mu\sigma}G_{\nu\lambda}) - \frac{2}{3}G_{\mu\nu}G_{\lambda\sigma}}{k^2 - m^2} T^{\lambda\sigma}(k)$$

conservation of energy and momentum implies $\partial_\mu T^{\mu\nu} = 0 \longrightarrow k_\mu T^{\mu\nu}(k) = 0$

$\longrightarrow$ replace $G_{\mu\nu}$ by $g_{\mu\nu}$

Looking at the interaction between two lumps of energy density T^{00}, we get

$$W(T) = -\frac{1}{2}\int \frac{d^4k}{(2\pi)^4} T^{00}(k)^* \frac{1 + 1 - \frac{2}{3}}{k^2 - m^2 + i\varepsilon} T^{00}(k)$$

As $(1 + 1 - \frac{2}{3}) > 0$, we see that masses attract each other!

§1.5 Feynman Diagrams

So far our quantum fields were composed of harmonic oscillators

→ let us now introduce anharmonic terms

$$Z[J] = \int \mathcal{D}\varphi \, e^{i\int d^4x \left(\frac{1}{2}[(\partial\varphi)^2 - m^2\varphi^2] - \frac{\lambda}{4!}\varphi^4 + J\varphi\right)}$$

anharmonic interaction

→ the above integral is not gaussian!

What should we do?

Feynman diagrams

Let's start with a simpler "baby problem":

$$Z[J] = \int_{-\infty}^{\infty} dq \, e^{-\frac{1}{2}m^2q^2 - \frac{\lambda}{4!}q^4 + Jq}$$

Let us series expand in λ

$$= \int_{-\infty}^{\infty} dq \, e^{-\frac{1}{2}m^2q^2 + Jq}\left[1 - \frac{\lambda}{4!}q^4 + \frac{1}{2}\left(\frac{\lambda}{4!}\right)^2 q^8 + \dots\right]$$

and integrate term by term.

This can be rewritten as:

$$Z[J]$$
$$= \left(1 - \frac{\lambda}{4!}\left(\frac{d}{dJ}\right)^4 + \frac{1}{2}\left(\frac{\lambda}{4!}\right)^2\left(\frac{d}{dJ}\right)^8 + \cdots\right)\int_{-\infty}^{\infty} dq\, e^{-\frac{1}{2}m^2q^2 + Jq}$$
$$= e^{-\frac{\lambda}{4!}\left(\frac{d}{dJ}\right)^4}\int_{-\infty}^{\infty} dq\, e^{-\frac{1}{2}m^2q^2 + Jq}$$
$$= \underbrace{\left(\frac{2\pi}{m^2}\right)^{\frac{1}{2}}}_{= Z[J=0,\, \lambda=0] =: Z[0,0]} e^{-\frac{\lambda}{4!}\left(\frac{d}{dJ}\right)^4} e^{\frac{1}{2m^2}J^2}$$

and define $\tilde{Z}[J] = Z[J]/Z[0,0]$

Suppose we want to compute the

(a) term of order λ and J^4 in $\tilde{Z}$.

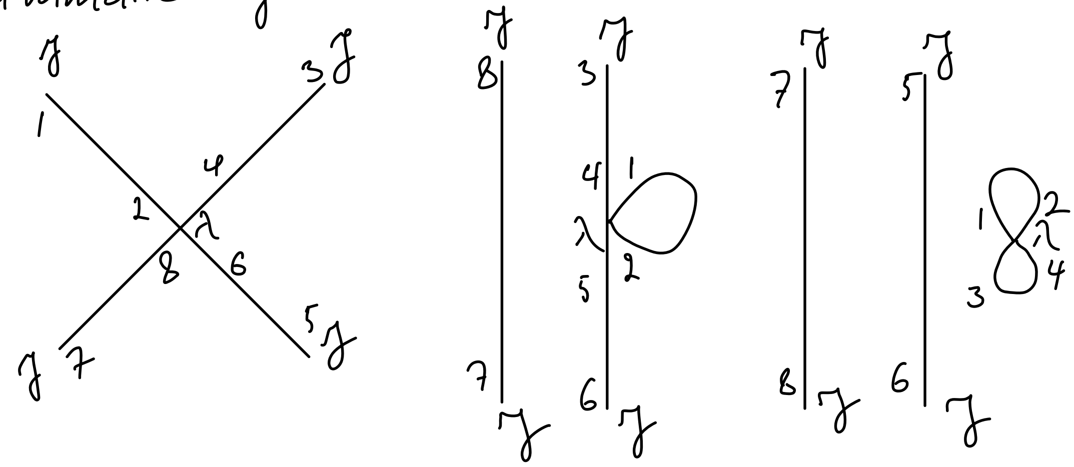
$\longrightarrow$ extract order J^8 term in $e^{J^2/2m^2}$, namely $\frac{1}{4!(2m^2)^4}J^8$, replace $e^{-\left(\frac{\lambda}{4!}\right)\left(\frac{d}{dJ}\right)^4}$ by $-\frac{\lambda}{4!}\left(\frac{d}{dJ}\right)^4$, and differentiate to get $\frac{8!\,(-\lambda)}{(4!)^3(2m^2)^4}J^4$

(b) another example: term of order λ^2 and J^6 is

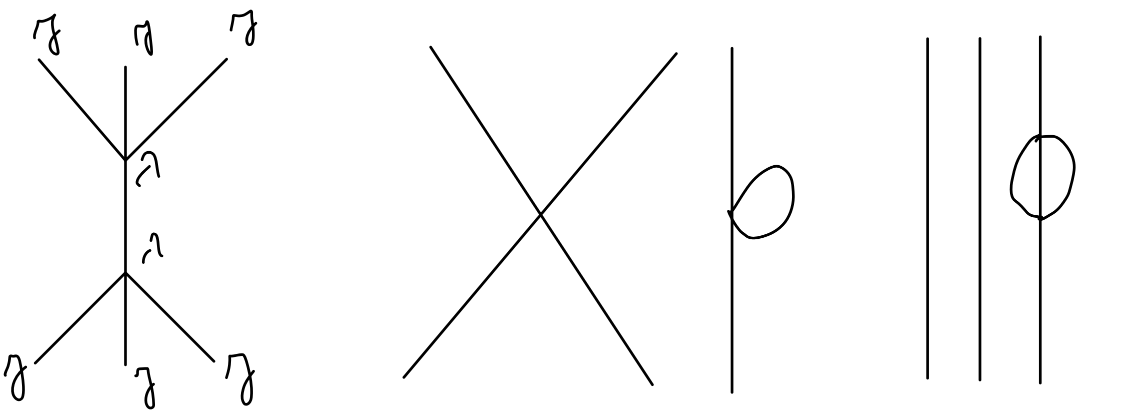
$$\frac{1}{2}\left(\frac{\lambda}{4!}\right)^2\left(\frac{d}{dJ}\right)^8\left(\frac{1}{7!(2m^2)^7}\right)J^{14} = \frac{14!\,(-\lambda)^2}{(4!)^2\, 6!\, 7!\, 2(2m^2)^7}J^6$$

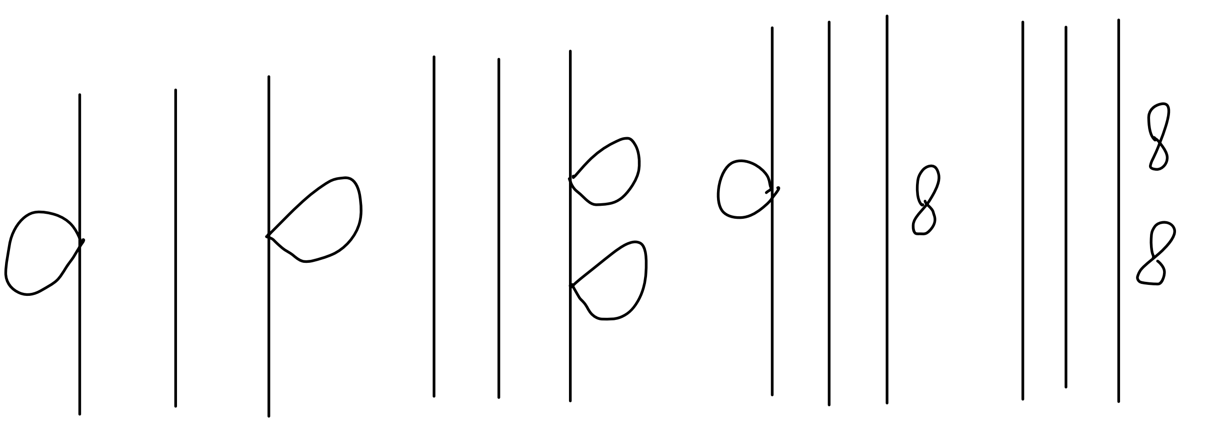
These calculations can be depicted diagrammatically:

(a) :



(b) :





→ rules:

1) for each vertex assign $-\lambda$
2) for each line $\frac{1}{m^2}$
3) for each external leg assign η

Wick contraction:

Alternatively, we can expand in powers of J:

$$Z[J] = \sum_{s=0}^{\infty} \frac{1}{s!} J^s \int_{-\infty}^{\infty} dq\, e^{-\frac{1}{2} m^2 q^2 - (\frac{\lambda}{4!}) q^4} q^s$$

$$= Z[0,0] \sum_{s=0}^{\infty} \frac{1}{s!} J^s G^{(s)}$$

the $O(\lambda)$ term in $G^{(4)}$ is

$$\frac{1}{(4!)^2} (-\lambda) \int_{-\infty}^{\infty} dq\, e^{-\frac{1}{2} m^2 q^2} q^8 = \frac{7!!}{(4!)^2} \frac{1}{m^8}$$

Connected versus disconnected

$$Z[J,\lambda] = Z[J=0,\lambda] e^{W[J,\lambda]}$$
$$= Z[J=0,\lambda] \sum_{N=0}^{\infty} \frac{1}{N!} [W(J,\lambda)]^N$$

diagrams with no external source

e.g. 

W is a sum of connected diagrams